

Planar Subgraph Isomorphism Revisited

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Abstract

The problem of SUBGRAPH ISOMORPHISM is defined as follows: Given a *pattern* H and a *host graph* G on n vertices, does G contain a subgraph that is isomorphic to H ? Eppstein [SODA 95, J’GAA 99] gives the first linear time algorithm for subgraph isomorphism for a fixed-size pattern, say of order k , and arbitrary planar host graph, improving upon the $O(n^{\sqrt{k}})$ -time algorithm when using the “Color-coding” technique of Alon et al [J’ACM 95]. Eppstein’s algorithm runs in time $k^{O(k)}n$, that is, the dependency on k is superexponential. We solve an open problem posed in Eppstein’s paper and improve the running time to $2^{O(k)}n$, that is, single exponential in k while keeping the term in n linear. Next to deciding subgraph isomorphism, we can construct a solution and count all solutions in the same asymptotic running time. We may enumerate ω subgraphs with an additive term $O(\omega k)$ in the running time of our algorithm. We introduce the technique of “embedded dynamic programming” on a suitably structured graph decomposition, which exploits the topology of the underlying embeddings of the subgraph pattern (rather than of the host graph). To achieve our results, we give an upper bound on the number of partial solutions in each dynamic programming step as a function of pattern size—as it turns out, for the planar subgraph isomorphism problem, that function is single exponential in the number of vertices in the pattern.

1 Introduction

In the literature, we often find results on polynomial time or even linear time algorithms for NP-hard problems. Take for example the NP-complete problem of computing an optimal tree-decomposition of a graph. Bodlaender [3] gives an algorithm in time $O(n)$ for this problem—restricted to input graphs of constant treewidth. The Graph Minor Theory developed by Robertson and Seymour implies amongst others that there is an $O(n^3)$ algorithm for the disjoint path problem, that is for finding disjoint paths between a constant number of terminals. Taking a closer look at such results, one notices that a function exponential in size of some constant c is hidden in the O -notation of the running time—here, c is the treewidth and the number of terminals, respectively. In another line of research, *parameterized complexity*, the primary goal is to rather find algorithms that minimize the exponential term of the running time. The first step here is to prove that such an algorithm with a separate exponential function exists, that is, that the studied problem is *fixed parameter tractable (FPT)* [13, 16, 21]. Such problem has an algorithm with time complexity bounded by a function of the form $f(k) \cdot n^{O(1)}$, where the *parameter function* f is a computable function only depending on k . The second step in the design of FPT-algorithms is to decrease the growth rate of the parameter function.

We can identify two different trends in which running times of exact algorithms are improved. First, one can decrease the degree of the polynomial term in the asymptotic running time, and second, one can focus on obtaining parameter functions with better exponential growth. In the present work, we achieve both goals for the computational problem PLANAR SUBGRAPH ISOMORPHISM.

SUBGRAPH ISOMORPHISM generalizes many important graph problems, such as HAMILTONICITY, LONGEST PATH, and CLIQUE. It is known to be *NP*-complete, even when restricted to planar graphs [18]. Until now, the best known algorithm to solve SUBGRAPH ISOMORPHISM, that is to find a subgraph of a given host graph isomorphic to a pattern H of order k (the number of vertices in H), is the naïve exhaustive search algorithm with running time $O(n^k)$ and no FPT-algorithm can be expected here [13]. For a pattern H of treewidth at most t , Alon et al. [1] give an algorithm of running time $2^{O(k)}n^{O(t)}$. For PLANAR SUBGRAPH ISOMORPHISM, given planar pattern and input graph, some considerable improvements have been made mostly during the 90's. The first improvement was provided by Plehn and Voigt [22], with running time $O(k^k)n^{O(\sqrt{k})}$. Using the elegant Color-coding technique of Alon et al. [1], one can devise an algorithm of running time $2^{O(k)}n^{O(\sqrt{k})}$. The current benchmark has been set by Eppstein [14] to $k^{O(k)}n$, by employing graph decomposition methods, similar to the Baker-approach [2] for approximating NP-complete problems on planar graphs. Eppstein's algorithm is actually the first FPT-algorithm for PLANAR SUBGRAPH ISOMORPHISM with k as parameter. Eppstein poses three open problems: a) whether one can extend the technique in [1] to improve the dependence on the size of the pattern from $k^{O(k)}$ to $2^{O(k)}$ for the decision problem of subgraph isomorphism; and whether one can achieve similar improvements b) for the counting version and c) for the listing version of the subgraph isomorphism problem.

Our results. In this work, we do not only achieve this single exponential behavior in k for all three problems—without applying the randomized coloring technique—we also keep the term in n linear. That is, we give an algorithm for PLANAR SUBGRAPH ISOMORPHISM for a pattern H of order k with running time $2^{O(k)}n$. Next to deciding subgraph isomorphism, we can construct a solution and count all solutions in the same asymptotic running time. We may list ω subgraphs with an additive term $O(\omega k)$ in the running time of our algorithm. Our algorithm also improves the time complexity of the previous approach [17] for patterns of size $k \in o(\sqrt{n} \log n)$.

The novelty of our result comes from *embedded dynamic programming*, a technique we find interesting on its own. Here, one decomposes the graph by separating it into induced subgraphs. In the dynamic programming step, one computes partial solutions for the separated subgraphs, that are updated to an overall solution for the whole graph. In ordinary dynamic programming, one would argue how the subgraph pattern hits separators of the host graph. Instead, in embedded dynamic programming for subgraph isomorphism, we proceed exactly the opposite way: we look at how separators can be routed through the subgraph pattern. As a consequence, we bound the number of partial solutions not by a function of the separator size of the host graph, but by a function of the pattern size—as it turns out, for the planar subgraph isomorphism problem, that function is single exponential in the number of vertices of the pattern. To obtain a good bound on the parameter function, we apply several fundamental enumerative combinatorics results in the technical sections of this work. Next to the number of cycles and face-vertex sequences in embedded graphs, these counting results give upper bounds on the number of planar triangulations and planar embeddings of the pattern.

Our algorithm is divided into two parts with the second part being the aforementioned embedded dynamic programming. For keeping the time complexity of our algorithm linear in the size of the host graph, we give a fast method for computing a graph decomposition with special properties: *Sphere-cut decompositions* are natural extensions of tree-decompositions to plane graphs, where the separator vertices are connected by a Jordan curve. In embedded dynamic programming we use sphere-cut decompositions with separators of size linearly bounded by the size of the subgraph pattern.

Theorem 1.1 *Let G be a planar graph on n vertices and H a pattern of order k . We can decide if there is a subgraph of G that is isomorphic to H in time $2^{O(k)}n$. We find subgraphs*

and count subgraphs of G isomorphic to H in time $2^{O(k)}n$ and enumerate ω subgraphs in time $2^{O(k)}n + O(\omega k)$.

It is worth mentioning that for k -LONGEST PATH on planar graphs, the authors of [12] give the first algorithm with time complexity subexponential in the parameter value. The algorithm has running time $2^{O(\sqrt{k})}n + O(n^3)$, employing the techniques *Bidimensionality* and topology-exploiting dynamic programming. *Bidimensionality Theory* employs results of Graph Minor Theory by Robertson and Seymour for planar graphs [23] and other structural graph classes to algorithmic graph theory (entry [6], for a survey [7]). Unfortunately, Bidimensionality does only work for finding specific patterns in a graph, such as k -paths, but not for subgraph isomorphism problems in general. For a survey on other planar subgraph isomorphism problems with restricted patterns, please consider [14].

Organization. After giving some definitions in Section 2, we show in Section 3 how to obtain a sphere-cut decomposition of small width. In Section 4 we restrict PLANAR SUBGRAPH ISOMORPHISM to PLANE SUBGRAPH ISOMORPHISM. We first give some technical lemmas in Section 4.1 to bound the number of ways a separator of the sphere-cut decomposition can be routed through a plane pattern. We describe and analyze embedded dynamic programming in Section 4.2 followed by subsuming the entire algorithm for PLANE SUBGRAPH ISOMORPHISM in Section 4.3. In Section 4 we bound the number of drawings of the pattern and show how to solve PLANAR SUBGRAPH ISOMORPHISM.

2 Preliminaries

Subgraph isomorphism. Let G, H be two graphs. We call G and H *isomorphic* if there exists a bijection $\nu : V(G) \rightarrow V(H)$ with $\{v, w\} \in E(G) \Leftrightarrow \{\nu(v), \nu(w)\} \in E(H)$. We call H *subgraph isomorphic to G* if there is a subgraph H' of G isomorphic to H .

Branch Decompositions. A *branch decomposition* $\langle T, \mu \rangle$ of a graph G consists of an unrooted ternary tree T (i.e., all internal vertices have degree three) and a bijection $\mu : L \rightarrow E(G)$ from the set L of leaves of T to the edge set of G . We define for every edge e of T the *middle set* $\text{mid}(e) \subseteq V(G)$ as follows: Let T_1 and T_2 be the two connected components of $T \setminus \{e\}$. Then let G_i be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of G_1 and G_2 , i.e., $\text{mid}(e) := V(G_1) \cap V(G_2)$. The *width* bw of $\langle T, \mu \rangle$ is the maximum order of the middle sets over all edges of T , i.e., $\text{bw}(\langle T, \mu \rangle) := \max\{|\text{mid}(e)| : e \in T\}$. An optimal branch decomposition of G is defined by a tree T and a bijection μ which together provide the minimum width, the *branchwidth* $\text{bw}(G)$.

Plane graphs and equivalent embeddings. Let Σ be the unit sphere. A *plane drawing* or *planar embedding* of a graph G with vertex set $V(G)$ and edge set $E(G)$ maps vertices to points in the sphere, and edges to simple curves between their end vertices, such that edges do not cross, except in common end vertices. A *plane graph* is a graph G together with a plane drawing. A *planar graph* is a graph that admits a plane drawing. For details, see e.g. [10]. The set of *faces* $F(G)$ of a plane graph G is defined as the union of the connected regions of $\Sigma \setminus G$. A subgraph of a plane graph G , induced by the vertices and edges incident to a face $f \in F(G)$, is called a *bound* of f . If G is 2-connected, each bound of a face is a cycle. We call this cycle *face-cycle* (for further reading, see e.g. [10]). For a subgraph H of a plane graph G , we refer to the drawing of G reduced to the vertices and edges of H as a *subdrawing* of G . Consider any two drawings G_1 and G_2 of a planar graph G . A *homeomorphism* of G_1 onto G_2 is a homeomorphism of Σ onto itself which maps vertices, edges, and faces of G_1 onto vertices,

edges, and faces of G_2 , respectively. We call two planar drawings of the same graph *equivalent*, if they are homeomorphic.

Theorem 2.1 (e.g. [10]) *Every 3-connected planar graph has a unique embedding in a sphere Σ up to homeomorphism.*

Triangulations. We call a plane graph G a *planar triangulation* or simply a *triangulation* if every face in $F(G)$ is bounded by a triangle (a cycle of length three). If H is a subdrawing of a triangulation G , we call G a *triangulation of H* .

Nooses and combinatorial nooses. A *noose* of a Σ -plane graph G is a simple closed curve in Σ that meets G only in vertices. From the Jordan Curve Theorem, it then follows that nooses separate Σ into two regions. Let $V(N) = N \cap V(G)$ be the vertices and $F(N)$ be the faces intersected by a noose N . The *length* of N is the number $|V(N)|$ of vertices in $V(N)$. The clockwise order in which N meets the vertices of $V(N)$ is a cyclic permutation π on the set $V(N)$.

Remark 2.2 *Let a plane graph H be a subdrawing of a plane graph G . Every noose N in G is also a noose in H and $V_H(N) \subseteq V_G(N)$.*

A *combinatorial noose* $N_C = [v_0, f_0, v_1, f_1, \dots, f_{k-1}, v_k]$ in a plane graph G is an alternating sequence of vertices and faces of G , such that

- f_i is a face incident to both v_i, v_{i+1} for all $i < k$,
- $v_0 = v_k$ and the vertices v_1, \dots, v_k are mutually distinct and
- if $f_i = f_j$ for any $i \neq j$ and $i, j = 0, \dots, k-1$, then the vertices v_i, v_{i+1}, v_j , and v_{j+1} do not appear in the order $(v_i, v_j, v_{i+1}, v_{j+1})$ on the bound of face $f_i = f_j$.

The *length* of a combinatorial noose $[v_0, f_0, v_1, f_1, \dots, f_{k-1}, v_k]$ is k .

Remark 2.3 *The order in which a noose N intersects the faces $F(N)$ and the vertices $V(N)$ of a plane graph G gives a unique alternating face-vertex sequence of $F(N) \cup V(N)$ which is a combinatorial noose N_C . Conversely, for every combinatorial noose N_C there exists a noose N with face-vertex sequence N_C .*

We may view combinatorial nooses as equivalence classes of nooses, that can be represented by the same face-vertex-sequence.

Sphere cut decompositions. For a Σ -plane graph G , we define a *sphere cut decomposition* or *sc-decomposition* $\langle T, \mu, \pi \rangle$ as a branch decomposition which for every edge e of T has a noose N_e that cuts Σ into two regions Δ_1 and Δ_2 such that $G_i \subseteq \Delta_i \cup N_e$, where G_i is the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$ and $T_1 \dot{\cup} T_2 = T \setminus \{e\}$. Thus N_e meets G only in $V(N_e) = \text{mid}(e)$ and its length is $|\text{mid}(e)|$. The vertices of every middle set $\text{mid}(e) = V(G_1) \cap V(G_2)$ are enumerated according to a cyclic permutation π on $\text{mid}(e)$.

The following two propositions will be crucial in that they give us upper bounds on the number of partial solutions we will compute in our dynamic programming approach. With both propositions, we will bound the number of combinatorial nooses in a plane graph by the number of cycles in the triangulation of some auxiliary graph. With the second proposition we bound the number of non-equivalent embeddings of planar graphs.

Proposition 2.4 ([4]) *No planar n -vertex graph has more than $2^{1.53n}$ simple cycles.*

Proposition 2.5 ([26]) *The number of non-isomorphic maximal planar graphs on n vertices is approximately $2^{3.24n}$.*

Proposition 2.5 also gives a bound on the number of non-isomorphic triangulations. Any embedding of a maximal planar graph G must be a triangulation, otherwise G would not be maximal. With Theorem 2.1, every maximal planar graph has a unique embedding which is a triangulation. On the other hand, every triangulated graph is maximal planar.

3 Computing sphere-cut decompositions in linear time

In this section we introduce an algorithm for computing sc-decompositions of bounded width. Let H be a connected subgraph of G with $|V(H)| = k$, and let $v \in V(H)$. Then H is a subgraph of the induced subgraph G^v of G , where $G^v = G[S]$ with $S = \{w \in S \mid \text{dist}(v, w) \leq k\}$ ($\text{dist}(v, w)$ denotes the length of a shortest path between v and w in G). This observation helps us to shrink the search space of our algorithm by cutting out chunks of G of bounded width and solve subgraph isomorphism separately on each chunk. With the algorithm of Tamaki [25], one can compute a branch decomposition of G^v of width $\leq 2k + 1$, following similar ideas as in the approach of Baker [2] for tree decompositions. With some simple modifications, we achieve the same result for sc-decompositions. In Appendix A we prove the following lemma and give an algorithm that computes a sc-decomposition of bounded width in linear time.

Lemma 3.1 *Let G be a plane graph with a rooted spanning tree whose root-leaf-paths have length at most k . We can find an sc-decomposition of width $2k + 1$ in time $O(kn)$.*

4 Plane subgraph isomorphism

In this section, we study the subgraph isomorphism problem on patterns and host graphs that are embedded in a sphere Σ . In Section 5 we carry over our results to planar graphs. We first introduce some topological tools that allow us to define a refined dynamic programming approach. At every step of the dynamic programming approach, we compute all possibilities of how a combinatorial noose N corresponding to a middle set of the sc-decomposition $\langle T, \mu, \pi \rangle$ of G can intersect a subdrawing equivalent to pattern H . Each intersection gives rise to a combinatorial noose of H . See Figure 1 for an illustration.

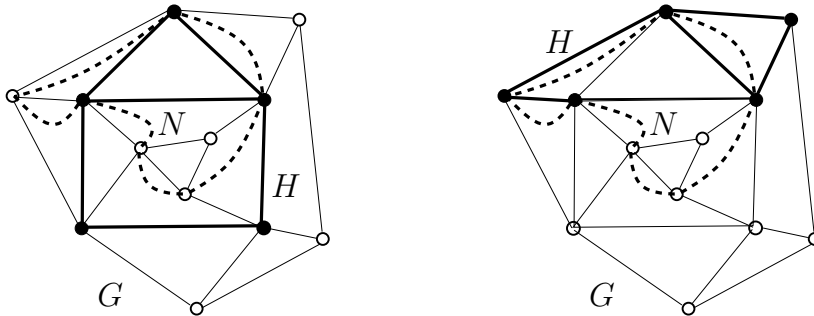


Figure 1: On the left, we have a plane graph G with an emphasized subdrawing H intersected by a combinatorial noose N indicated by dashed lines. On the right, we have the same graph G with a different copy of H intersected by N .

The running time of the algorithm crucially depends on the number of combinatorial nooses in H . The aim of this section is to prove the following:

Theorem 4.1 *Let G be a plane graph on n vertices and H be a plane graph on $k \leq n$ vertices. We can decide if there is a subdrawing of G that is equivalent to H in time $2^{O(k)}n$. We can find*

and count subdrawings equivalent to H in time $2^{O(k)}n$, and enumerate ω subdrawings in time $2^{O(k)}n + O(\omega k)$.

4.1 Combinatorial nooses in plane graphs

For a refined algorithm analysis we now take a close look at combinatorial nooses of plane graphs. In particular we are interested in counting the number of combinatorial nooses. In this subsection, we will prove the following lemma:

Lemma 4.2 *Every plane k -vertex graph has $2^{O(k)}$ combinatorial nooses.*

Before proving this lemma, we show that every combinatorial noose of a plane graph on k vertices corresponds to a cycle in some other plane graph on at most $O(k)$ vertices. First we relate combinatorial nooses in a planar triangulation H' to the cycles in H' . In a second step we relate combinatorial nooses of a 3-connected plane graph H to cycles in the triangulations of H . Finally, we will show that for any plane graph H there is an auxiliary graph H^* , such that the combinatorial nooses of H can be injectively mapped to the cycles of the triangulations of H^* . From Proposition 2.4 we know an upper bound on the number of cycles in planar graphs, which we employ to prove Lemma 4.2.

Lemma 4.3 *Let H be a planar triangulation and $N_C = [v_0, f_0, v_1, f_1, \dots, f_{k-1}, v_k]$ a combinatorial noose of H . Then for every pair of consecutive vertices v_i, v_{i+1} in N_C , there is an edge v_i, v_{i+1} in $E(H)$. That is, the sequence $[v_0, v_1, \dots, v_k]$ is a simple cycle in H if $|V(N_C)| > 2$, and if $|V(N_C)| = 2$, it corresponds to a single edge in H .*

Proof. Since H is triangulated, we have that every $f_i \in N_C$ is bounded by a triangle Δ where v_i, v_{i+1} are two of the three vertices of Δ and v_i, v_{i+1} have an edge in common. Since vertices occur only once in N_C , f_i is unique in N_C if $|V(N_C)| > 2$, that is, there is no $f_j \in N_C$ with $i \neq j$ and $f_i = f_j$. Hence we map each f_i one-to-one to edge $e_i = \{v_i, v_{i+1}\}$ and get a cycle $[v_0, e_0, v_1, e_1, \dots, e_{k-1}, v_k]$. For $|V(N_C)| = 2$, f_0 and f_1 are incident faces to edge $\{v_0, v_1\}$. For an illustration, see Figure 2. \square

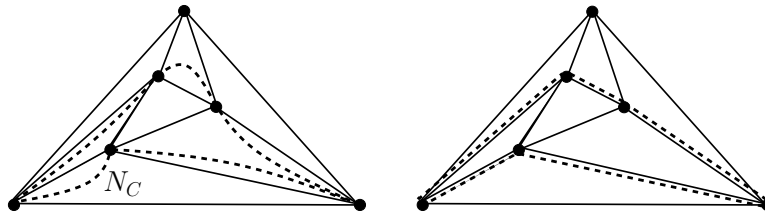


Figure 2: On the left, there is a triangulation with a combinatorial noose N_C indicated by dashed lines. On the right, we have mapped the noose to a cycle indicated by dashed lines.

Lemma 4.4 *Let H be a 3-connected plane graph and $N_C = [v_0, f_0, v_1, f_1, \dots, f_{k-1}, v_k]$ a combinatorial noose of H with $|V(N_C)| > 2$. Then there exists a planar triangulation H' of H , such that $[v_0, v_1, \dots, v_k]$ is a cycle in H' .*

Proof. We proceed in two phases. First we iteratively add edges to H and transform N_C into another combinatorial noose such that every two consecutive vertices in N_C have a common edge. Then we triangulate the resulting graph.

For every pair of consecutive vertices v_i, v_{i+1} in N_C , if v_i, v_{i+1} have no edge in common, add $e_i = \{v_i, v_{i+1}\}$ to $E(H)$. Thereby the drawing of e_i splits f_i into two new faces f_i^a and

f_i^b , bounded by face-cycle C^a and C^b respectively, where $C^a \cap C^b = e_i$. Since N_C corresponds to a noose by Remark 2.3 and nooses are not self-intersecting, we observe the following for $|V(N_C)| > 2$: for every $f_j = f_i$ in N_C with $j \neq i$ we have that both v_j, v_{j+1} are in one of C_a and C_b . Thus, adding edge e_i will not cross any other edge added in this process. In N_C , we replace f_i by one of f_i^a and f_i^b , and every $f_j = f_i$ by f_i^a if F_j is bounded by C^a and by f_i^b otherwise. Once we have an edge for every pair of consecutive vertices in N_C , we note that for every sub-sequence $[v_i, f_i, v_{i+1}]$ of N_C the edge $e_i = \{v_i, v_{i+1}\}$ is incident to face f_i since, by 3-connectivity, edge e_i is uniquely embedded in H . We then add edges arbitrarily to obtain a triangulation H' . By Lemma 4.3, the vertices of N_C correspond to a cycle in H' . For an illustration, see Figure 3. \square

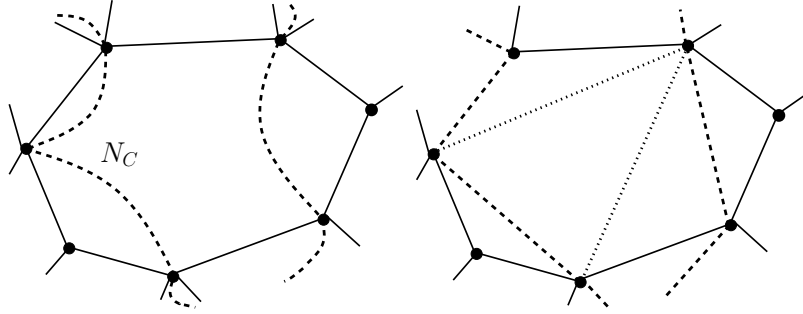


Figure 3: On the left, we see a face of our plane graph with a combinatorial noose N_C indicated with dashed lines. To the right, we have mapped the noose to a cycle indicated with dashed lines and dotted lines indicating the face triangulation.

If H is not 3-connected, a problem may occur in the last step of the previous proof when triangulating H . Consider a sub-sequence $[v_i, f_i, v_{i+1}]$ in N_C . We assume there already exists an edge $e_i = \{v_i, v_{i+1}\}$ and v_i, v_{i+1} separate H , that is, H is 2-connected. Then it may be the case that e_i is not incident to f_i , and thus, any triangulation of H has an edge crossing N_C . We surpass this problem in the general case by triangulating some auxiliary graph instead. For an edge $e = \{v, w\}$ of a graph H we *subdivide* e by adding a vertex u to $V(H)$ and replacing e by two new edges $e_1 = \{v, u\}$ and $e_2 = \{u, w\}$. In a drawing of H , we place point u in the middle of the drawing of e partitioning e into e_1 and e_2 .

Lemma 4.5 *Let H be plane graph and $N_C = [v_0, f_0, v_1, f_1, \dots, f_{k-1}, v_k]$ a combinatorial noose of H with $|V(N_C)| > 2$. Let H^* be obtained by subdividing every edge in $E(H)$. There exists a planar triangulation H' of H^* such that $[v_0, v_1, \dots, v_k]$ is a cycle in H' .*

Proof. The combinatorial noose N_C is a combinatorial noose in H^* , too. As for any two consecutive vertices v_i, v_{i+1} in N_C there is no edge in H^* and each vertex in N_C is unique, we may add edges to H^* as in the proof of Lemma 4.4 and triangulate H^* . \square

Proof of Lemma 4.2. If H is triangulated, we have with Lemma 4.3 that every combinatorial noose corresponds to a unique cycle in H . By Proposition 2.4, the number of cycles in H is bounded by $2^{1.53k}$. Since for every edge of a cycle in H , we have two choices for a combinatorial noose to visit an incident face, we get the overall upper bound of $2^{2.53k}$ on the number of combinatorial nooses. If H is plane, we have to count the triangulations either of H (Lemma 4.4) or of H^* (Lemma 4.5). By Proposition 2.5 and the comments below it, there are at most $2^{3.24\ell}$ non-isomorphic triangulations on ℓ vertices. Let us denote this set of triangulated graphs by Φ . We note that H (resp. H^*) is a subgraph of some graph of Φ , say of all graphs in $\Phi_H \subseteq \Phi$ with $|\Phi_H| \geq 1$. Since every triangulated graph is 3-connected, we have with Theorem 2.1 that

every graph H' in Φ_H has a unique embedding in Σ up to homeomorphism. The plane graph H (resp. H^*) is then a subdrawing of a drawing equivalent to an arbitrary plane embedding of H' in Σ . Thus, the number of triangulations times the number of combinatorial nooses in each triangulation is an upper bound on the number of combinatorial nooses in H , here $2^{5.77k}$ (resp. in H^* , here $2^{9.77k}$). \square

For embedded dynamic programming on a sc-decomposition $\langle T, \mu, \pi \rangle$, we can argue with Remark 2.2 that if H is a subdrawing of G , then noose N formed by the middle set $\text{mid}(e)$ is a noose of H , too. Recalling Remark 2.3, the alternating sequence of vertices and faces of H visited by N forms a combinatorial noose N_C in H .

This observation allows us to discuss the results from a combinatorial point of view without the underlying topological arguments. Instead of nooses we will refer to combinatorial nooses in the remaining section.

4.2 Embedded dynamic programming

In embedded dynamic programming, the basic difference to usual dynamic programming is that we do not check for every partial solution for a given problem if or how it lies in the graph processed so far. Instead, we check how the graph that we have processed so far is intersecting the entire solution, that is how the graph is *embedded* into our solution. For subgraph isomorphism, we compute every possible way the processed subdrawing G_{sub} of G is embedded in the plane pattern H up to homeomorphism, subject to how the bound of G_{sub} intersects H . This bound is a combinatorial noose N separating G_{sub} from the rest of G . The number of solutions we get is bounded by the number of combinatorial nooses in H we can map N onto. We describe the algorithm in what follows.

Dynamic programming. We root sc-decomposition $\langle T, \mu, \pi \rangle$ at some node $r \in V(T)$. For each edge $e \in T$, let L_e be the set of leaves of the subtree rooted at e . The subgraph G_e of G is induced by the edge set $\{\mu(v) \mid v \in L_e\}$. The vertices of $\text{mid}(e)$ form a combinatorial noose N that separates G_e from the residual graph.

Assuming H is a subdrawing of G , the basic idea of embedded dynamic programming is that we are interested in how the vertices of the combinatorial noose N are intersecting faces and vertices of H . Since every noose in G is a noose in H , we can map N to a combinatorial noose N^H of H , bounding (clockwise) a unique subgraph H_{sub} of H .

In each step of the algorithm, all solutions for a sub-problem in G_e are computed, namely all possibilities of how N is mapped onto a combinatorial noose N^H in H that separates H_{sub} from the rest of H , where $H_{\text{sub}} \subseteq H$ is isomorphic to subgraphs of G_e . For every middle set, we store this information in an array. It is updated in a bottom-up process starting at the leaves of $\langle T, \mu, \pi \rangle$. During this updating process it is guaranteed that the ‘local’ solutions for each subgraph associated with a middle set of the sc-decomposition are combined into a ‘global’ solution for the overall graph G .

Step 0: Initializing the middle sets. Let G be a plane graph with a rooted sc-decomposition $\langle T, \mu, \pi \rangle$ and let H be a plane pattern. For every middle set $\text{mid}(e)$ of $\langle T, \mu, \pi \rangle$ let N be the associated combinatorial noose in G with face-vertex sequence of $F(N) \cup V(N)$. Let \mathfrak{L} denote the set of all combinatorial nooses of H whose length is at most the length of N . We now want to map N order preserving to each $N^H \in \mathfrak{L}$. We map vertices of N to both vertices and faces of H . Therefore, we consider partitions of $V(N) = V_1(N) \dot{\cup} V_2(N)$ where vertices in $V_1(N)$ are mapped to vertices of $V(H)$ and vertices in $V_2(N)$ to faces of $F(H)$. We define a mapping $\gamma : V(N) \cup F(N) \rightarrow V(H) \cup F(H)$ relating N to the combinatorial nooses in \mathfrak{L} . For every $N^H \in \mathfrak{L}$ on faces and vertices of set $F(N^H) \cup V(N^H)$ and for every partition $V_1(N) \dot{\cup} V_2(N)$ of $V(N)$ mapping γ is valid if

- a) γ restricted to $V_1(N)$ is a bijection to $V(N^H)$;
- b) for every $v \in V_1(N)$ we have $\gamma(v) \in V(N^H)$, and for every $v \in V_2(N)$ we have $\gamma(v) \in F(N^H)$;
- c) for every $f \in F(N)$ we have $\gamma(f) \in F(N^H)$;
- d) for every pair $v_h, v_j \in V(N)$ such that $[\gamma(v_h), f, \gamma(v_j)]$ is a subsequence of N^H for a face $f \in F(N^H)$ and for every vertex $v_i \in V(N)$ with v_i lying inbetween v_h and v_j in the sequence N , we have $\gamma(v_i) = f$;
- e) for every $v_i \in V(N)$ and subsequence $[f_{i-1}, v_i, f_i]$ of N , if $\gamma(v_i) \in F(N^H)$, we have $\gamma(f_{i-1}) = \gamma(v_i) = \gamma(f_i)$;
- f) for every pair of vertices w_i, w_j in $V(N^H)$: if $\{w_i, w_j\} \in E(H)$ then $\{\gamma^{-1}(w_i), \gamma^{-1}(w_j)\} \in E(G)$.

Items a) to c) say where to map the faces and vertices of N to. Items d) and e) make sure that if two vertices v_h, v_j in sequence $N = [\dots, v_h, \underline{\dots}, v_j, \dots]$ are mapped to two vertices w_i, w_{i+1} that appear in sequence N^H as $[\dots, w_i, f_i, w_{i+1}, \dots]$ then every face and vertex inbetween v_h, v_j in sequence N (here underlined) is mapped to face f_i . Item f) rules out the invalid solutions, that is, we do not map a pair of vertices in G that have no edge in common to the endpoints of an edge in H . We do so because if H is a subdrawing of G then an edge in H is an edge in G , too. For an illustration, see Figure 4.

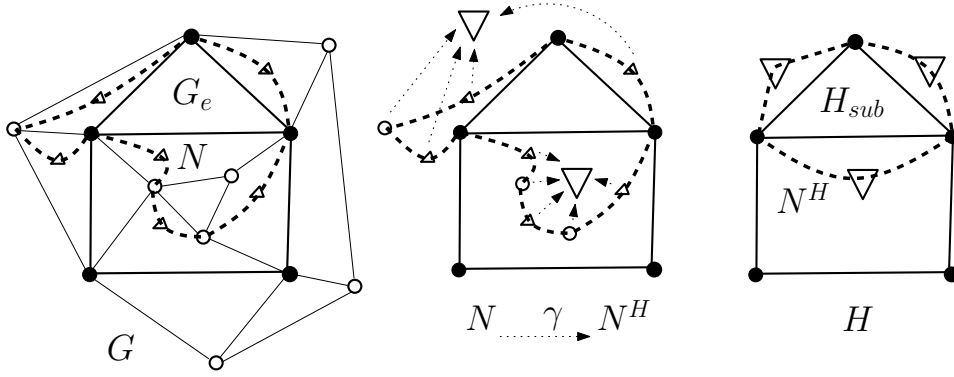


Figure 4: On the left, we have a plane graph G with a subdrawing H emphasized. A combinatorial noose N separating subgraph G_e is indicated by dashed lines. The vertices of N are full and empty circles and the faces triangles. In the middle, we have H and indicate to which faces (big triangles) of H vertices and faces of N are mapped by γ . This gives us combinatorial noose N^H on the right, separating subgraph H_{sub} .

We assign an array A_e to each $\text{mid}(e)$ consisting of all tuples $\langle N^H, \gamma(N) \rangle$ each representing a valid mapping $\gamma(N)$ from combinatorial noose N corresponding to $\text{mid}(e)$ to a combinatorial noose $N^H \in \mathfrak{L}$. The vertices and faces of N are oriented clockwise around G_e . Without loss of generality, we assume for every $\langle N^H, \gamma(N) \rangle \in A_e$ the orientation of N^H to be clockwise around the subgraph H_{sub} of H isomorphic to a subgraph of G_e .

Step 1: Update process. We update the arrays of the middle sets in post-order manner from the leaves of T to root r . In each dynamic programming step, we compare the arrays of two middle sets $\text{mid}(e), \text{mid}(f)$ in order to create a new array assigned to the middle set $\text{mid}(g)$, where e, f and g have a vertex of T in common. From [12] we know about a special property of sc-decompositions: namely that the combinatorial noose N_g is formed by the symmetric difference of the combinatorial nooses N_e, N_f and that $G_g = G_e \cup G_f$. In other words, we are ensured that if two solutions on G_e and G_f bounded by N_e and N_f *fit together*, then they form a new solution on G_g bounded by N_g . We now determine when two solutions represented as

tuples in the arrays A_e and A_f fit together. We update two tuples $\langle N_e^H, \gamma_e(N_e) \rangle \in A_e$ and $\langle N_f^H, \gamma_f(N_f) \rangle \in A_f$ to a new tuple in A_g if

- for all $v \in V(N_e) \cap V(N_f)$, $\gamma_e(v) = \gamma_f(v)$;
- for all $f \in F(N_e) \cap F(N_f)$, $\gamma_e(f) = \gamma_f(f)$;
- for the subgraph H_e of H separated by N_e^H and the subgraph H_f of H separated by N_f^H , we have that $E(H_e) \cap E(H_f) = \emptyset$ and $V(H_e) \cap V(H_f) \subseteq \{\gamma(v) \mid v \in V(N_e) \cap V(N_f)\}$.

If N_e and N_f fit together, we get a valid mapping $\gamma_g : N_g \rightarrow N_g^H$ as follows:

- for every $x \in (V(N_e) \cup F(N_e)) \cap (V(N_f) \cup F(N_f)) \cap (V(N_g) \cup F(N_g))$ we have $\gamma_e(x) = \gamma_f(x) = \gamma_g(x)$;
- for every $y \in (V(N_e) \cup F(N_e)) \setminus (V(N_f) \cup F(N_f))$ we have $\gamma_e(y) = \gamma_g(y)$;
- for every $z \in (V(N_f) \cup F(N_f)) \setminus (V(N_e) \cup F(N_e))$ we have $\gamma_f(z) = \gamma_g(z)$.

We have that γ_g is a valid mapping from N_g to the combinatorial noose N_g^H that bounds subgraph $H_g = H_e \cup H_f$. Thus, we add tuple $\langle N_g^H, \gamma_g(N_g) \rangle$ to array A_g .

Step 2: End of DP If, at some step, we have a solution where the entire subgraph H is formed, we exit the algorithm confirming. That is, if $H = H_e \cup H_f$ and H_i is bounded by N_i (for both $i \in \{e, f\}$) then the combinatorial noose N_g is bounding the subgraph of G isomorphic to H . We are able to output this subgraph by reconstructing the solution top-down in $\langle T, \mu, \pi \rangle$. If at root r no subgraph isomorphic to H has been found, we output 'FALSE'.

Correctness of DP Let plane graph H be a subdrawing of G . We have seen already in Step 0 how we map every combinatorial noose of G that identifies a separation of G via a valid mapping γ to a combinatorial noose of H determining a separation of H . Every edge of H is bounded by a combinatorial noose N^H of length two, which is determined by tuple $\langle N^H, \gamma(N) \rangle$ in an array assigned to a leaf edge of T . We need to show that Step 1 computes a valid solution for N_g from N_e and N_f for incident edges e, f, g . We note that the property that the symmetric difference of the combinatorial nooses N_e and N_f forms a new combinatorial noose N_g is passed on to the combinatorial nooses N_e^H, N_f^H and N_g^H of H , too. If the two solutions fit together, then H_e of H separated by N_e^H and subgraph H_f of H separated by N_f^H only intersect in the image of $V(N_e) \cap V(N_f)$. We may observe that N_e^H and N_f^H intersect in a continuous alternating subsequence with order reversed to each other, i.e., $N_e^H|_{N_e \cap N_f} = \overline{N_f^H}|_{N_e \cap N_f}$, where $\overline{N^H}$ means the reversed sequence N^H . Since every oriented N^H identifies uniquely a separation of $E(H)$, we can easily determine if two tuples $\langle N_e^H, \gamma_e(N_e) \rangle \in A_e$ and $\langle N_f^H, \gamma_f(N_f) \rangle \in A_f$ fit together and form a new subgraph of H . If H is a subdrawing of G , then at some step we will enter Step 2 and produce the entire H .

Running time analysis. We first give an upper bound on the size of each array. The number of combinatorial nooses in \mathfrak{L} we are considering is bounded by the total number of combinatorial nooses in H , which is $2^{O(|V(H)|)}$ by Lemma 4.2. The number of partitions of vertices of any combinatorial noose N is bounded by $2^{|V(N)|}$. Since the order of both N^H and N is given we only have $2|V(H)|$ possibilities to map vertices of N to N^H , once the vertices of N are partitioned. Thus, in an array A_e we may have up to $2^{O(|V(H)|)} \cdot 2^{|V(N)|} \cdot |V(H)|$ tuples $\langle N_e^H, \gamma_e(N_e) \rangle$. We first create all tuples in the arrays assigned to the leaves. Since middle sets of leaves only consist of an edge in G , we get arrays of size $O(|V(H)|^2)$ which we compute in the same asymptotic running time. When updating middle sets $\text{mid}(e), \text{mid}(f)$, we compare every tuple of one array A_e to every tuple in array A_f to check if two tuples fit together. We can compute the unique subgraph H_e (resp. H_f) described by a tuple in A_e (resp. A_f), compare two tuples in A_e, A_f and create a new tuple in A_g in time linear in the order of $V(N)$ and $V(H)$. Since the size of A_g is bounded by $2^{O(|V(H)|)} \cdot 2^{O(|V(N)|)}$, the update process for two middle sets takes the same

asymptotic time. Assuming sc-decomposition $\langle T, \mu, \pi \rangle$ of G has width ω and $|V(H)| \leq \omega$, we get the following result.

Lemma 4.6 *For a plane graph G with a given sc-decomposition $\langle T, \mu, \pi \rangle$ of G of width w and a plane pattern H on $k \leq w$ vertices we can search for a subdrawing of G equivalent to H in time $2^{O(w)} \cdot n$.*

4.3 The algorithm

We present the overall algorithm for solving PLANE SUBGRAPH ISOMORPHISM with running time stated in Theorem 4.1.

Algorithm 4.1: Plane subgraph isomorphism: PLSI.

Input : Plane graph G ; Plane pattern H of order k .

- 1 Choose an arbitrary vertex v in G .
 - 2 Partition $V(G)$ into $S_0 \cup S_1 \cup \dots \cup S_\ell$ with $S_i = \{w \in V(G) : \text{dist}(v, w) = i\}$
 - 3 **for** every $G_i = G[S_i \cup \dots \cup S_{i+k}]$ with $0 \leq i \leq \ell - k$ **do**
 - 4 Compute sc-decomposition $\langle T, \mu, \pi \rangle$ of G_i .
 - 5 Do embedded dynamic programming on $\langle T, \mu, \pi \rangle$ to find a subgraph of G_i isomorphic to H and intersecting S_i .
-

Partitioning the vertex set in Line 1 of Algorithm 4.1 PLSI, is a similar approach to the well-known Baker-approach [2]. Every vertex set S_i contains the vertices of distance i to the chosen vertex v . $S_0 = \{v\}$ and ℓ is the maximum distance in G from v . The graph G_i in Line 1 is induced by the sets S_i, \dots, S_{i+k} . As in [14], we may argue that every vertex in G appears in at most k subgraphs G_i . This keeps our running time linear in n . We can apply Lemma 3.1 to each G_i in Line 1 to compute sc-decomposition $\langle T, \mu, \pi \rangle$ of width $\leq 2k$, by adding a root vertex r for the BFS tree and make r adjacent to every vertex in S_i . The dynamic programming approach can easily be turned into an algorithm counting subgraph isomorphisms (similar to [14]), by using a counter in the dynamic programming. Using an inductive argument, for every subgraph G_i in Line 1 we only compute subgraphs intersecting with vertices in S_i and thus omit double-counting. We can also adopt our technique to list the subgraphs of G isomorphic to H .

5 Planar subgraph isomorphism

Now we consider the case when both pattern H and host graph G are planar but not embedded. However, we observe that if H is isomorphic to a subgraph of G , then for every planar embedding of G there exists a drawing of H that is equivalent to a subdrawing of G . Hence, we may simply embed G planarly, and run the algorithm of the previous section for all non-equivalent embeddings of H . The following lemma tells us that the number of times we call the algorithm is restricted, too.

Lemma 5.1 *Every planar k -vertex graph has $2^{O(k)}$ non-equivalent embeddings in Σ .*

Proof. By Proposition 2.5, there are at most $2^{3 \cdot 24k}$ non-isomorphic maximal planar graphs on k vertices. Every planar graph H is a subgraph of a maximal planar graph. Every maximal planar graph has a unique embedding which is a triangulation. Thus, every embedding of H is a subdrawing of a triangulation of H . The number of such subgraphs is bounded by the number of edge subsets of H' , since for every edge subset of $S \subseteq E(H')$ of same cardinality as $E(H)$, $H'[S]$ may be isomorphic to H . In this case, $H'[S]$ then gives a possible embedding of H in Σ . Hence, the number of embeddings of H in Σ up to homeomorphism is bounded by $2^{6 \cdot 24k}$. \square

Algorithm 5.1: Planar subgraph isomorphism.

Input : Planar graph G , Planar pattern H of size k .

Compute a planar embedding of G .

if H *triangulated or 3-connected* **then** Return PLSI(G, H).

for *every non-equivalent embedding I of H* **do**

 Return PLSI(G, I).

The whole algorithm We compute in Algorithm 5.1 every non-equivalent embedding of H using the constructive proof of Lemma 5.1. That is, we compute the set \mathcal{H} of non-isomorphic maximal planar graphs in time proportional to its size using the algorithm in [20]. For every graph $H' \in \mathcal{H}$ and every subdrawing I of H' we check whether I is isomorphic to H by using the linear time algorithm for planar graph isomorphism in [19]¹. By Lemma 5.1, we then call Algorithm 4.1 $2^{O(k)}$ times, for each plane drawing I isomorphic to H . This ensures us that Algorithm 5.1 has running time as stated in Theorem 1.1.

6 Conclusion

We have shown how to use topological graph theory to improve the results on the already mentioned variations of PLANAR SUBGRAPH ISOMORPHISM, solving the open problems posed in [14] and [12]. With the results of [15], [14] extends the feasible graph class from planar graphs to apex-minor-free graphs. This cannot be done with the tools presented here. However, [11] devise a truly subexponential algorithm for k -LONGEST PATH in H -minor-free graphs and thus apex-minor-free graphs, employing the structural theorem of Robertson and Seymour [24] and the results of [8, 5, 9]. Can the structure of H -minor-free graphs, be exploited for our purposes?

It seems unlikely that our work can be extended to obtain a subexponential algorithm. The first reason, mentioned in the introduction, is that Bidimensionality applies to subgraphs with minor properties rather than to general subgraphs. Secondly, our enumerative bounds are either tight or of lower bound $2^{\Omega(k)}$. We want to pose the open problem: Is PLANE SUBGRAPH ISOMORPHISM solvable in time $2^{o(k)}n^{O(1)}$?

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¹We get a list of embeddings of H , from which we can delete equivalent drawings by a modification of the algorithm in [19]—namely isomorphism test for face-vertex graphs.

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A SC-Decompositions in linear time

For a plane graph G we define a *radial graph* R_G as follows: R_G is a bipartite graph with the bipartition $F(G) \cup V(G)$. A vertex $v \in V(G)$ is adjacent in R_G to a vertex $f \in F(G)$ if and only if the vertex v is incident to the face f in the drawing of G .

Let G be a plane graph with some vertex $r \in V(G)$ and R_G its radial graph. Let T be a spanning tree of G rooted at r that is determined by breadth first search. Choose a face f adjacent to root r . If the longest path from r to a leaf of T is ℓ then the distance d_f in the radial graph R_G from vertex f to any other (face)vertex x is at most $2\ell + 1$. This is due to the fact that there exists an edge $\{f, r\}$ in R_G , and for every edge in T there is a detour in R_G of at most two edges. [25] show how to obtain a branch decomposition of width d_f out of a BFS spanning tree rooted at r of the radial graph². Set f to be the outer face of G . Let T be a BFS spanning tree of R_G rooted at f and let ℓ be the maximum distance in T from r to a leaf. We give now a compact presentation of the algorithm of [25] and show that it translates to constructing a sphere-cut decomposition of G . We define *contracting* a vertex v as identifying all vertices of $N(v)$ to a single vertex and deleting v .

In [25], Algorithm A.1 is proved to compute a branch decomposition of planar graph G of width $2\ell + 1$ in time $O(\ell n)$.

Claim A.1 *Algorithm A.1 computes a sc-decomposition of G .*

Proof. Observe that the edge set T^* of the dual graph of R_G (the so-called medial graph) minus the dual edges of T_S in Line 3 forms a tree due to the acyclicity of T_S . Every node of T^* corresponds to an edge of G and in fact, spans the edge set of G . For turning T^* into a branch-decomposition we a) bijectively map the leafs of T^* to the edges of G and b) make T^* ternary. For a) we generate for every node v in T^* from Line 3–3 one ternary tree (or single edge tree), a *local tree* C_v with v one leaf. In Line 3–3 those local trees are merged from the leaves of T^* to its root in post order such that each C_v contributes to leaf v in the overall ternary tree C_r . We show now that the such formed branch decomposition actually obeys our definition of an sc-decomposition, that is the vertices in each middle set form a cycle in the radial graph. Note that every edge of G forms a 4-cycle in R_G . Let $e^* \in E(T^*)$ be the dual edge

²In fact the authors construct a carving decomposition out of the spanning tree of the dual graph of the radial graph that one obtains after deleting the dual edges of the BFS spanning tree.

Algorithm A.1: Computing SC-decomposition.

Input : Plane graph G , face $f \in F(G)$, radial graph R_G .

Output: Branch-decomposition of G of width at most $2\ell + 1$.

- 1 Construct embedded BFS tree T_S of R_G at root f .
 - 2 Set $T^* = R_G^* \setminus E(T_S)^*$ the dual graph of R_G without the edge set dual to T_S
 - 3 **for** every node v in T^* **do**
 - 4 **if** $\deg(v)_T = 1$ **then**
 - 5 create C_v a single edge, with nodes labeled $\{v\}$ and $\{N(v)\}$;
 - 6 **else**
 - 7 create embedded ternary tree C_v with $|N(v)| + 1$ leaves;
 - 8 label one leaf with $\{v\}$ and the other leaves with $N(v)$ keeping a clockwise order.
 - 9 *(in post order)* **for** every edge $\{u, v\}$ in T^* , where v is the parent node **do**
 - 10 combine C_u and C_v by identifying leaf $\{v\}$ in C_u with leaf $\{u\}$ in C_v , and
 - 11 contract the identified node and set new tree to be named C_v .
 - 12 **Return** $(C_r$ (for r root of T^*)).
-

of edge $e = \{f, g\} \in E(R_G \setminus T_S)$. Then the union of e and the path through T_S from f over the lowest common ancestor of f, g in T_S to g forms a cycle in R_G that separates the two subtrees of T^* that are separated by e^* . Thus, T^* already possesses middle sets that form nooses in G . However T^* is not ternary since it may have maximum node degree 4. The leaves of each local tree C_v in Line 3 are embedded in the same order as the inverse of their labels, the neighboring nodes of v in T^* , and thus we keep the same ordering in the overall ternary tree C_r . Every edge e of C_r comes from an edge of one of the local trees C_v , and e separates the neighbors $N_1(v)$ from $N_2(v)$ where the disjoint union $N_1(v), N_2(v)$ form the neighborhood $N(v)$ of v in T^* . Like this, T^* falls apart into two subtrees each bounded by a cycle in R_G formed similarly as above by the union of $N_i(v)$ bounding minimal path in T_S and the path through the edges of T^* induced by $N_i[v]$, for $i = 1$ and $i = 2$ respectively.